UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 7TH FEBRUARY 2013

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

http://www.ukmt.org.uk

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates. More comprehensive solutions are on the website.

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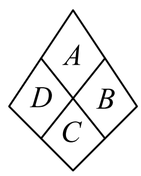
1. **D** In order to be a multiple of 6, a number must be both even and a multiple of 3. Of the numbers given, only B 999 998 and D 999 996 are even. Using the rule for division by 3, we see that, of these two, only 999 996 is a multiple of 3.

2. **B** 180 000 eggs per hour is equivalent to 3000 eggs per minute, i.e. to 50 eggs per second.

3. **E** The figure is itself a quadrilateral. It can be divided into four small quadrilaterals labelled A, B, C, D. There are also four quadrilaterals formed in each case by joining together two of the smaller quadrilaterals: A and B; B and C; C and D; D and A.



4. **D** The number of seeds in a special packet is $1.25 \times 40 = 50$. So the number of seeds which germinate is $0.7 \times 50 = 35$.

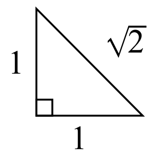
5. **E** A wheatear travels the distance of almost 15 000 km in approximately 50 days. This is on average roughly 300 km per day.

6. **E** In order, the values of the expressions given are: $1 - 0 = 1$; $2 - 1 = 1$; $9 - 8 = 1$; $64 - 81 = -17$; $625 - 1024 = -399$.

7. A Only two colours are needed for the upper four faces of the octahedron. If, for example, blue and red are used then these four faces may be painted alternately red and blue. Consider now the lower four faces: every face adjacent to an upper blue face may be painted red and every face adjacent to an upper red face may be painted blue. So only two colours are required for the whole octahedron.

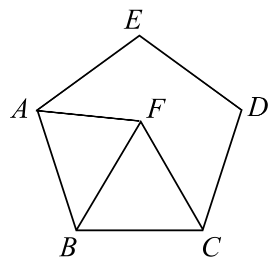
8. D Let the number of scores of 1 be n. Then the product of the scores is $1^n \times 2 \times 3 \times 5 = 30$. Therefore $1 \times n + 2 + 3 + 5 = 30$, i.e. $n = 20$. So Jim threw 23 dice.

9. A Let the length of the shorter sides of the cards be 1 unit. Then, by Pythagoras' Theorem, the length of the hypotenuse of each card is $\sqrt{1^2 + 1^2} = \sqrt{2}$.



So the lengths of the perimeters of the five figures in order are: $4\sqrt{2}$; $4 + 2\sqrt{2}$; $4 + 2\sqrt{2}$; 6; $4 + 2\sqrt{2}$. Also, as $\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2\frac{1}{4} > 2$ we see that $\frac{3}{2} > \sqrt{2}$. Therefore, $4\sqrt{2} < 6 < 4 + 2\sqrt{2}$. So figure A has the shortest perimeter.

10. C The sum of the interior angles of a pentagon is 540° so $\angle ABC = 540° \div 5 = 108°$. Each interior angle of an equilateral triangle is 60°, so $\angle FBC = 60°$.

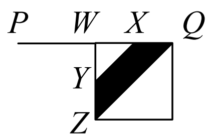


Therefore $\angle ABF = 108° - 60° = 48°$. As $ABCDE$ is a regular pentagon, $BC = AB$. However, $BC = FB$ since triangle BFC is equilateral.
So triangle ABF is isosceles with $FB = AB$.
Therefore $\angle FAB = \angle AFB = (180° - 48°) \div 2 = 66°$.

11. C We first look at $66 = 2 \times 3 \times 11$. Its factors involve none, one, two or all three of these primes. So the factors are 1, 2, 3, 11, 6, 22, 33, 66; and their sum is $144 = 12^2$. Similarly, we can check that the sum of the factors of 3, 22, 40 and 70 is, respectively, $4 = 2^2$, $36 = 6^2$, 90 and $144 = 12^2$. So 40 is the only alternative for which the sum of the factors is not a square number.

12. D As the words ‘three’ and ‘five’ contain 5 and 4 letters respectively, their ‘sum’ will be a 9-letter word. Of the alternatives given, only ‘seventeen’ contains 9 letters.

13. B The diagram shows the top-right-hand portion of the square. The shaded trapezium is labelled $QXYZ$ and W is the point at which ZY produced meets PQ.



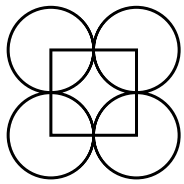
As $QXYZ$ is an isosceles trapezium, $\angle QZY = \angle ZQX = 45°$.
Also, as YX is parallel to ZQ, $\angle XYW = \angle WXY = 45°$. So WYX and WZQ are both isosceles right-angled triangles. As $\angle ZWQ = 90°$ and Z is at the centre of square $PQRS$, we deduce that W is the midpoint of PQ. Hence $WX = XQ = \frac{1}{4}PQ$. So the ratio of the side-lengths of similar triangles WYX and WZQ is 1 : 2 and hence the ratio of their areas is 1 : 4.
Therefore the area of trapezium $QXYZ = \frac{3}{4} \times$ area of triangle $ZWQ = \frac{3}{32} \times$ area $PQRS$ since triangle ZWQ is one-eighth of $PQRS$. So the fraction of the square which is shaded is $4 \times \frac{3}{32} = \frac{3}{8}$.

14. D As all the fractions are raised to the power 3, the expression which has the largest value is that with the largest fraction in the brackets.
Each of these fractions is a little larger than $1\frac{1}{2}$. Subtracting $1\frac{1}{2}$ from each in turn, we get the fractions $\frac{1}{14}, \frac{1}{6}, \frac{1}{4}, \frac{3}{10}, 0$, the largest of which is $\frac{3}{10}$ (because $0 < \frac{1}{14} < \frac{1}{6} < \frac{1}{4} = \frac{2\frac{1}{2}}{10} < \frac{3}{10}$). Hence $\left(\frac{9}{5}\right)^3$ is the largest.

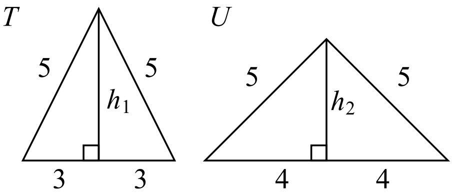
15. B From the information given, we may deduce that the number of coins is a multiple of each of 3, 5, 7. Since these are distinct primes, their lowest common multiple is $3 \times 5 \times 7 = 105$. So the number of coins in the bag is a multiple of 105. So there are 105 coins in the bag since 105 is the only positive multiple of 105 less than or equal to 200.

16. A The image of a straight line under a rotation is also a straight line. The centre of rotation, the point (1, 1), lies on the given line and so also lies on the image. The given line has slope 1 and so its image will have slope -1. Hence graph A shows the image.

17. D The radius of each disc in the figure is equal to half the side-length of the square, i.e. $\frac{1}{\pi}$. Because the corners of a square are right-angled, the square hides exactly one quarter of each disc.
So three-quarters of the perimeter of each disc lies on the perimeter of the figure. Therefore the length of the perimeter is $4 \times \frac{3}{4} \times 2\pi \times \frac{1}{\pi} = 6$.

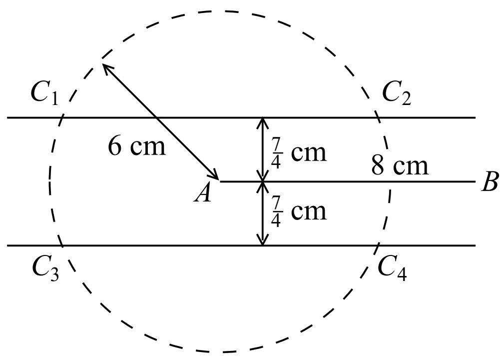


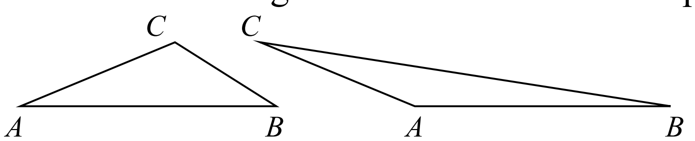
18. C The diagrams show isosceles triangles T and U. The perpendicular from the top vertex to the base divides an isosceles triangle into two congruent right-angled triangles as shown in both T and U. Evidently, by Pythagoras' Theorem, $h_1 = 4$ and $h_2 = 3$. So both triangles T and U consist of two '3, 4, 5' triangles and therefore have equal areas.



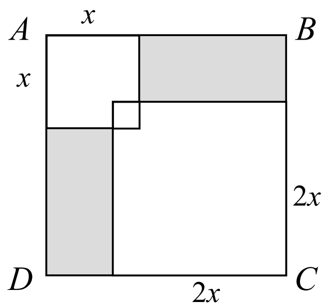
19. E $(x \div (y \div z)) \div ((x \div y) \div z) = \left(x \div \frac{y}{z}\right) \div \left(\left(\frac{x}{y}\right) \div z\right) = \left(x \times \frac{z}{y}\right) \div \left(\frac{x}{y} \times \frac{1}{z}\right)$
$= \frac{xz}{y} \div \frac{x}{yz} = \frac{xz}{y} \times \frac{yz}{x} = z^2$.

20. B Let the base AB of the triangle be the side of length 8 cm and let AC be the side of length 6 cm. So C must lie on the circle with centre A and radius 6 cm as shown. The area of the triangle is to be 7 cm^2, so the perpendicular from C to AB (or to BA produced) must be of length $\frac{7}{4}$ cm.
The diagram shows the four possible positions of C. However, since $\angle BAC_1 = \angle BAC_3$ and $\angle BAC_2 = \angle BAC_4$, these correspond to exactly two possibilities for the length of the third side AC. The diagrams below show the two possibilities.

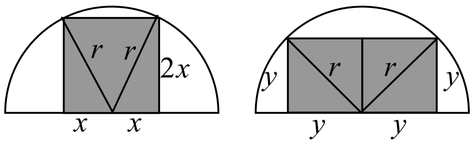




21. B The large square has area $196 = 14^2$. So it has side-length 14. The ratio of the areas of the inner squares is 4 : 1, so the ratio of their side-lengths is 2 : 1. Let the side-length of the larger inner square be $2x$, so that of the smaller is x. The figure is symmetric about the diagonal AC and so the overlap of the two inner squares is also a square which therefore has side-length 1. Thus the vertical height can be written as $x + 2x - 1$. Hence $3x - 1 = 14$ and so $x = 5$. Also, the two shaded rectangles both have side-lengths $2x - 1$ and $x - 1$; that is 9 and 4. So the total shaded area is 72.

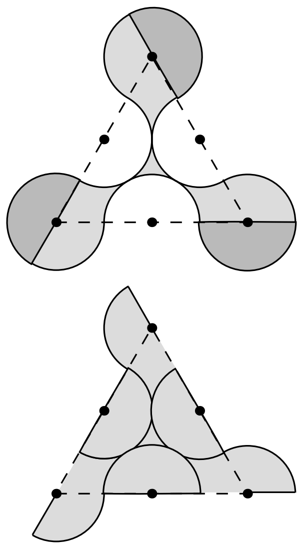


22. D Let the radius of each semicircle be r. In the left-hand diagram, let the side-length of the square be $2x$. By Pythagoras' Theorem, $x^2 + (2x)^2 = r^2$ and so $5x^2 = r^2$. So this shaded area is $4x^2 = \frac{4r^2}{5}$. In the right-hand diagram, let the side-length of each square be y. Then by Pythagoras' Theorem, $y^2 + y^2 = r^2$ and so this shaded area is r^2. Therefore the ratio of the two shaded areas is $\frac{4}{5} : 1 = 4 : 5$.



23. A If Alfred is telling the truth, the other three are lying (as their statements would then be false) and we know this is not the case. Hence Alfred is lying. Similarly, if Horatio is telling the truth, the other three are lying which again cannot be the case. So Horatio is lying. Hence the two who are telling the truth are Bernard and Inigo. (A case where this situation would be realised would be if the brothers in descending order of age were Alfred , Bernard, Horatio and Inigo.)

24. B The length of the side of the triangle is equal to four times the radius of the arcs. So the arcs have radius $2 \div 4 = \frac{1}{2}$. In the first diagram, three semicircles have been shaded dark grey. The second diagram shows how these semicircles may be placed inside the triangle so that the whole triangle is shaded. Therefore the difference between the area of the shaded shape and the area of the triangle is the sum of the areas of three sectors of a circle. The interior angle of an equilateral triangle is 60°, so the angle at the centre of each sector is $180° - 60° = 120°$. Therefore each sector is equal in area to one-third of the area of a circle. Their combined area is equal to the area of a circle of radius $\frac{1}{2}$. So the required area is $\pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.



25. D $\left(10^{640} - 1\right)$ is a 640-digit number consisting entirely of nines. So $\frac{\left(10^{640} - 1\right)}{9}$ is a 640-digit number consisting entirely of ones.

Therefore $\frac{10^{641} \times \left(10^{640} - 1\right)}{9}$ consists of 640 ones followed by 641 zeros. So $\frac{10^{641} \times \left(10^{640} - 1\right)}{9} + 1$ consists of 640 ones followed by 640 zeros followed by a single one. Therefore it has 1281 digits.

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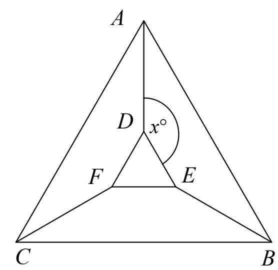
1. **A** 25% of $\frac{3}{4} = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$.

2. **D** The first four options are the smallest positive integers which are both odd and not prime. However, the next largest odd numbers after 9, 15, 21 are 11, 17, 23 respectively and these are all prime. The next largest odd number after 25 is 27, which is not prime. So 25 is the smallest positive integer which satisfies all three conditions.

3. **E** Clearly AD lies along one of the lines of symmetry of the figure. So $\angle FDA = \angle EDA = x°$. Triangle DEF is equilateral so $\angle EDF = 60°$.
The angles which meet at a point sum to 360°, so $x + x + 60 = 360$.
Therefore $x = 150$.



4. **C** Since m is even, $m = 2k$ for some integer k. So $3m + 4n = 2(3k + 2n)$; $5mn = 2(5kn)$; $m^3n^3 = 8k^3n^3$ and $5m + 6n = 2(5k + 3n)$, which are all even. As n is odd, $3n$ is also odd. So $m + 3n$ is an even integer plus an odd integer and is therefore odd. The square of an odd integer is odd so $(m + 3n)^2$ is odd.